Interpretation of the Surface Renewal Model Through the Prandtl Mixing Length Theory

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The surface renewal model (*1, 2*) for turbulent mixing processes has been shown by Toor and Marchello (*3*) to be applicable to turbulent pipe flow. From experimental considerations, Toor and Marchello conclude that the mixing coefficient S is related to the fluid dynamics and geometry of the problem by

$$S \propto \frac{U}{D} \cdot [N_{Re_D}]^{0.8} \qquad (1)$$

The mixing coefficient S represents the rate of renewal of macroscopic fluid lumps at the wall surface, which is brought about by turbulent mixing in the fluid.

For the turbulent flow of a fluid over a flat surface, one would expect the mixing or renewal between the fluid bulk state and the wall to be controlled by the velocity fluctuations normal to the wall (v^1) and the Prandtl mixing length (l). We may therefore write

$$S \propto \frac{|v^1|}{l} \qquad (2)$$

From Prandtl's mixing length theory (*4*), we know that

$$\left.\begin{aligned} |v^1| &= \text{constant} \cdot |u^1| \\ \text{and} \qquad l\frac{d\bar{u}}{dy} &= u^1 \end{aligned}\right\} \qquad (3)$$

Substitution for Equation (3) in Equation (2) yields

$$S \propto \frac{d\bar{u}}{dy} \qquad (4)$$

Now, $(d\bar{u})/(dy)$ in Equation (4) is considered at the wall, since the turbulent exchange is between the bulk flow and the wall. At the wall $(d\bar{u})/(dy) = (\tau_0 g_c)/(\mu)$, and τ_0 for the turbulent flow over a flat surface can be expressed as

$$\tau_0 = \frac{f}{2} \cdot \rho \frac{U^2}{g_c}$$

and $f/2 = \text{constant} \cdot [N_{Re_x}]^{-0.2}$. Hence, $(g_c\tau_0)/(\mu) = \text{constant}.\ U/x \cdot [N_{Re_x}]^{0.8}$. Therefore,

$$S \propto \frac{U}{x} \cdot [N_{Re_x}]^{0.8}$$

By interpretating the mixing coefficient S through the Prandtl mixing length theory, the dependence of S on the fluid dynamics and geometry involved in the exchange process has been shown for the turbulent flow over a flat surface. Extension to the pipe flow problem of (*3*) can readily be carried out.

NOTATION

D = tube internal diameter, L
$\left(\frac{f}{2}\right)$ = friction factor, dimensionless
g_c = gravitational constant, dimensionless
l = Prandtl mixing length, L
N_{Re_D}; N_{Re_x} = Reynolds number based on tube diameter and surface length, respectively, dimensionless
S = mixing or renewal coefficient, $1/t$
$\bar{u}$ = mean flow velocity parallel to flat surface, L/t
u^1 = instantaneous velocity fluctuation of the flow in direction parallel to flat surface, L/t
v^1 = instantaneous velocity fluctuation of the flow in direction perpendicular to flat surface, L/t
x = distance along flat surface from the leading edge, L
y = distance perpendicular to flat surface, L
ρ = fluid density, M/L^3
μ = fluid absolute viscosity, M/Lt
τ_0 = shear stress at wall surface, M/Lt^2

LITERATURE CITED

1. Danckwerts, P. V., *Ind. Eng. Chem.*, 43, 1460 (1951).
2. ———, *AIChE J.*, 1, 457 (1954).
3. Toor, H. L., and J. M. Marchello, *ibid.*, 4, 97 (1958).
4. Schlichting, H., "Boundary Layer Theory," 6 ed., McGraw-Hill, New York (1968).

A Criterion for Fully Developed Flow of Polymer Melts in a Circular Tube

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Capillary rheometers have been used to evaluate the rheological properties of all types of fluids for many years. In order for the data obtained from such instruments to be meaningful, it is necessary that fully developed flow be achieved within the capillary. In the case of nonelastic fluids, the entrance length required for the attainment of fully developed flow is generally accepted as being

$$L_e = cDN_{Re} \qquad (1)$$

The value of c, as given by various authors, ranges between 0.02 and 0.06.

On the basis of this result, it was for a long time believed that the entrance length for polymer melts would be quite small, since in most cases melt flow occurs at very low Reynolds numbers. Work done on the determination of melt die swell in recent years indicates that this is not the case. On the basis of this work, it has become recognized that the entrance length required for

polymer melts lies somewhere in the range of 15 to 150 (*1* to *3*).

Data recently obtained by the present authors in their study of the exit pressure phenomenon tends to reinforce the contention that the entrance length for polymer melts is very much greater than that of nonelastic fluids and is, for both polyethylene and polypropylene, around 20 capillary diam.

BACKGROUND

It was first observed by Sakiadis (*4*) and then by Arai et al. (*5*) and Mori and Funatsu (*6*) that if one plots the total normal stress in the radial direction at the capillary wall $s_{rr}(R,z)$ as a function of the axial position z, one obtains a plot such as the one shown in Figure 1. This plot is referred to as a *pressure profile*. The important points to notice here are that the plot becomes linear at a small L/D ratio and that the extrapolation of the plot to the exit of the capillary shows that the radial normal stress at the exit $S_{rr}(R,L)$ is greater than atmospheric.

In general, the stress $S_{rr}(R,L)$ is compressive and is therefore, by convention, taken to be negative. As a matter of convenience one defines the exit pressure as

$$P_{R,L} = -S_{rr}(R,L) \qquad (2)$$

It should be noted that in the case of polymer melts, the exit pressure is approximately equal to the primary normal stress difference (*7*).

The authors have designed apparatus that permits an accurate determination of the exit pressure and have used it to study, among other things, the effect of the L/D ratio on the exit pressure. It should also be noted that the authors have found that Lodge's hole pressure hypothesis (*8*) does not appear to be valid in the case of capillary flow of polymer melts (*7*).

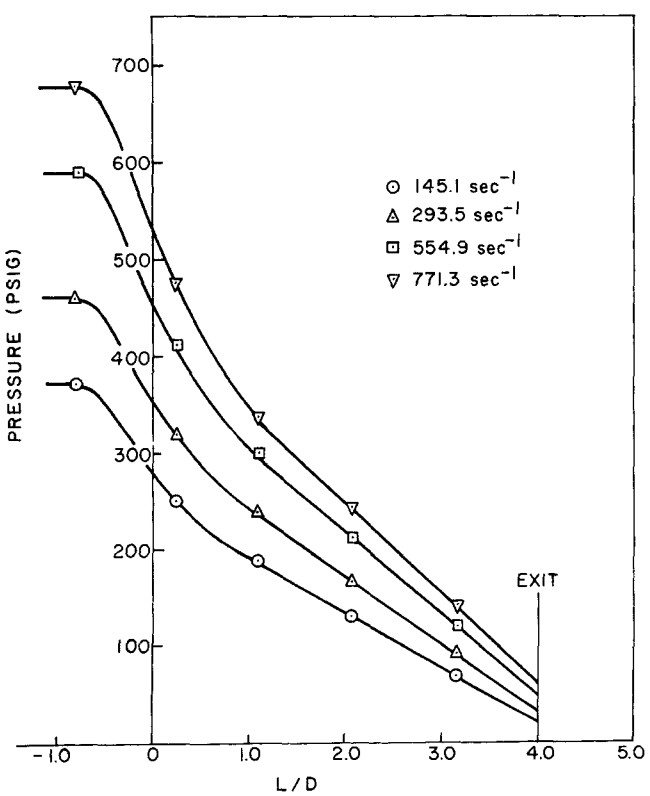

Fig. 1. Pressure profiles for high density polyethylene at 180°C. ($L/D = 4$; $D = 0.125$ in.)

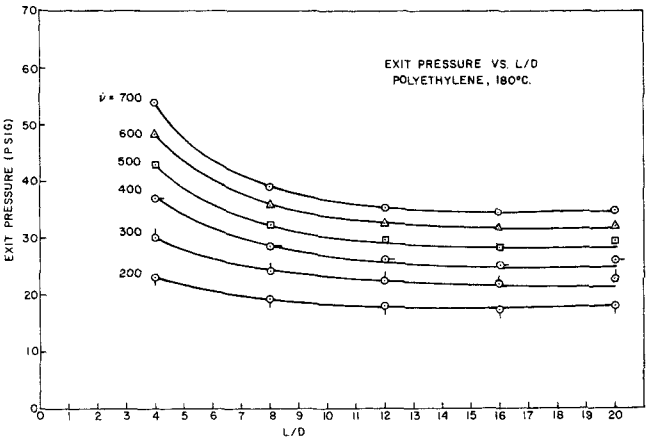

Fig. 2. Exit pressure vs. *L/D* ratio for high density polyethylene.

RESULTS AND DISCUSSION

Typical pressure profiles for polyethylene are given in Figure 1. These are typical of all of the results obtained in that they exhibit two distinct portions. The first, which corresponds to the flow region which extends from the capillary entrance to a length equivalent to about 1 capilliary diam. into the capillary, shows distinct evidence of a developing flow. Beyond this length, the pressure profile becomes linear. If one accepts the constancy of the pressure gradient as being sufficient proof of fully developed flow, then one is led to the conclusion that the entrance length for polymer melts is equivalent to about 1 capillary diam.

It is clear from the currently held theories regarding viscoelasticity that the elastic behavior of a viscoelastic fluid, such as a polymer melt, is directly related to the characteristics of the flow field, and so long as the field is changing, so also will the elastic behavior. If, then, the criterion for fully developed flow stated above is valid, the elastic response of the fluid should not be a function of the L/D ratio for ratios greater than 1 or 2.

In Figure 2 the exit pressures of polyethylene are plotted as functions of the L/D ratio. From these it is clear that the elastic response [it will be recalled that for polymer melts the exit pressure is approximately equal to the primary normal stress difference (*7*)] does not become constant until the L/D ratio reaches a value of about 20. It is therefore clear that the flow is not fully developed at an L/D ratio of 1 or 2.

To corroborate this result, measurements of the polyethylene die swell ratio were obtained at various L/D ratios. The results of this work are given in Figure 3, in which the die swell ratio is plotted as a function of the

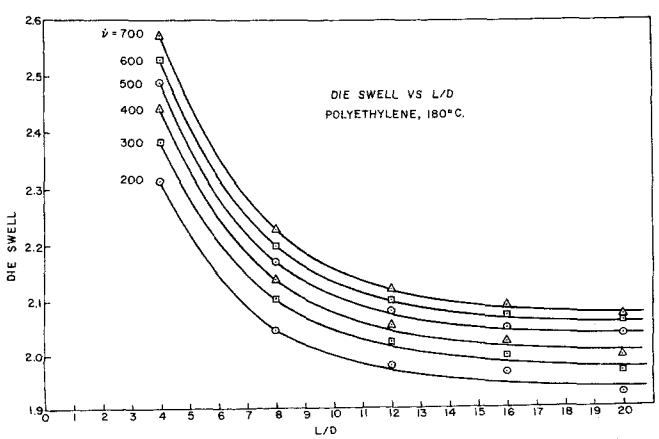

Fig. 3. Die swell ratio vs. *L/D* ratio for high density polyethylene.

L/D ratio. It can again be seen that an L/D ratio of about 20 is required to insure constancy of the die swell ratio. The similarity in behavior of both the die swell and the exit pressure is to be expected in light of the fact that both are manifestations of the same elastic nature. It should be noted that the value of 20 diam. for the entrance length agrees quite favorably with the value of 15 found by Bagley (*3*) for another polyethylene, but not with that found by Metzner et al. (*1*).

CONCLUSION

On the basis of both die swell and exit pressure measurements, one is led to the conclusion that the entrance length required for the fully developed flow of polymer melts is quite large. It is further evident that the measurement of the pressure gradient (with currently available melt pressure transducers) is not sufficiently sensitive to respond to changes in the flow field that occur at distances from the entrance which are greater than that equivalent to about 1 capillary diam. Measurement of elastic properties such as exit pressure and die swell are apparently much better indicators of the development of the flow pattern. Of these, the die swell appears to be more sensitive and so should be used in the determination of the entrance length.

NOTATION

c = constant in Equation (1)
D = capillary diameter
L_e = entrance length
N_{Re} = Reynolds number
$P_{R,L}$ = exit pressure
$S_{rr}(R,L)$ = the total normal stress in the radial direction at the exit

LITERATURE CITED

1. Metzner, A. B., E. L. Carley, and I. K. Park, *Modern Plastics*, **37**, No. 11, 133 (1960).
2. Bagley, E. B., S. H. Storey, and D. C. West, *J. Appl. Polymer Sci.*, **7**, 1661 (1963).
3. Kowalski, R. C., Ph.D. dissertation, Polytechnic Inst., Brooklyn, New York (1963).
4. Sakiadis, B. C., *AIChE J.*, **8**, 317 (1962).
5. Arai, Teikichi, paper presented at the Fifth International Congress of Rheology, Kyoto, Japan (Oct., 1968).
6. Mori, Yoshiro, and Kazumori Funatsu, *Chemistry of High Polymers (Japan)*, **25**, 391 (1968).
7. Han, C. D., Marvin Charles, and Wladimir Philippoff, *Trans. Soc. Rheol.*, to be published.
8. Lipson, J. M., and A. S. Lodge, *Rheologica Acta*, **7**, 364 (1968).

In Defense of the Crank-Nicolson Method

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A new finite-difference technique for solving the heat conduction equation has recently been published by Liu (*1*). To demonstrate the effectiveness of his method, Liu applied it to the problem $\partial u/\partial t = \partial^2 u/\partial x^2$ subject to the initial condition $u(x, 0) = 0$, $0 \leq x \leq 1$ and the boundary conditions $\partial u(0, t)/\partial x = 0$, $u(1, t) = 1$, $t > 0$. For $\Delta x = 0.05$ and $\Delta t = 0.002$, corresponding to $\lambda = \Delta t/(\Delta x)^2 = 0.8$, the results computed for u were compared with the known analytical solutions and also with those obtained by the Crank-Nicolson (C-N) method, identified by Liu as "the classical implicit method." By taking the case that is the most favorable to Liu, the error involved at $t = 0.5$ by using his "average" method is less by two or three orders of magnitude than the error involved using the C-N method, *according to Liu* (see lines 1 and 2 of Table 1).

This communication emphasizes that Liu's paper has, in any event, portrayed an unduly pessimistic picture of the accuracy of the C-N method and, furthermore, that an almost trivial modification of the C-N method (scheme B below) yields results just as accurate as those generated by Liu's method.

TABLE 1. $10^5 \times$ ERROR AT $t = 0.5$, WITH $\Delta x = 0.05$

	Method	λ	$x = 0.2$	$x = 0.6$	$x = 0.8$
1	C-N (Liu)	0.8	1,767.0	844.6	414.0
2	Liu (average)	0.8	6.9	3.3	0.1
3	C-N (scheme A)	0.8	−91.5	−56.6	−29.8
4	C-N (scheme B)	0.8	−4.2	−2.7	−1.4
5	C-N (scheme B)	5.0	−3.0	−2.1	−0.6

I have reprogrammed the C-N method for the same problem, using standard techniques [see (*2*), for example] with single-precision arithmetic on an IBM-360/67 computer. The question that arises, almost invariably, is what value of u should be used in the approximation at $x = 1$ and $t = 0$. Strictly speaking (scheme A), according to the above mathematical statement of the initial and boundary conditions, the value $u = 0$ should be taken (as was the case in Liu's paper), followed thereafter by $u = 1$ at $x = 1$ for $t > 0$. Alternatively, as a numerical experiment (scheme B), one can also follow what is perhaps the more standard practice and try $u = 1$ at $x = 1$ for $t \geq 0$. There are other possibilities, of course, since we are dealing in terms of finite-difference *approximations*.

The results of my computations are summarized in lines 3, 4, and 5 of Table 1. The error is defined here as the numerical approximation minus the exact value, which may differ in sign from Liu's definition as "the difference between the exact solution and the numerical solution."

My conclusions are evident: that the C-N errors according to Liu's programming are inordinately large, even compared with the results of Scheme A, and that with the slight modification according to scheme B, the C-N method gives results at least as good as those obtained by Liu's method, *even* (at this particular value of time) with a highly inflated time step.

LITERATURE CITED

1. Liu, S.-L., *AIChE J.*, **15**, 334-338 (1969).
2. Carnahan, B., H. A. Luther, and J. O. Wilkes, "Applied Numerical Methods," Wiley, New York (1969).